18.06 MIDTERM 2 - SOLUTIONS

PROBLEM 1

(1) Use the Gram-Schmidt process to convert the vectors:

$$\begin{bmatrix} 4 \\ 6 \\ 12 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 \\ -9 \\ -4 \end{bmatrix}$$

into an orthonormal basis of the vector space they span. **Show all your steps!** *(15 pts)*

Solution: The first step is to rescale $\boldsymbol{v}_1 = \begin{bmatrix} 4 \\ 6 \\ 12 \end{bmatrix}$ to that it has length 1:

$$\boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{||\boldsymbol{v}_1||} = \frac{\boldsymbol{v}_1}{14} = \frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$$

The next step is to modify $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -9 \\ -4 \end{bmatrix}$ so that its orthogonal to $\boldsymbol{q}_1$:

$$\boldsymbol{w}_2 = \boldsymbol{v}_2 - \mathrm{proj}_{\boldsymbol{q}_1}\boldsymbol{v}_2 = \boldsymbol{v}_2 - \boldsymbol{q}_1(\boldsymbol{q}_1 \cdot \boldsymbol{v}_2) = \boldsymbol{v}_2 + 7 \cdot \boldsymbol{q}_1 = \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$$

The final step is to rescale $\boldsymbol{w}_2$ so that it has length 1:

$$\boldsymbol{q}_2 = \frac{\boldsymbol{w}_2}{||\boldsymbol{w}_2||} = \frac{\boldsymbol{w}_2}{7} = \frac{1}{7}\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$$

(2) Use the previous part to obtain a factorization:

$$\boxed{A = QR} \qquad \text{of the matrix} \qquad A = \begin{bmatrix} 4 & 1 \\ 6 & -9 \\ 12 & -4 \end{bmatrix}$$

where Q has orthonormal columns and R is an upper triangular square matrix. **Show all the steps of your argument, and explain how it derives from part (1)!** *(10 pts)*

Solution: The matrix Q is precisely:

$$Q = \left[\ \boldsymbol{q}_1 \mid \boldsymbol{q}_2 \ \right] = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ 3 & -6 \\ 6 & 2 \end{bmatrix}$$

As for R, we will compute it by recasting the Gram-Schmidt process as multiplying A on the right by elimination and diagonal matrices:

$$AD_1^{\left(\frac{1}{14}\right)} E_{12}^{(7)} D_2^{\left(\frac{1}{7}\right)} = Q$$

By inverting the D and E matrices and moving them to the right, we get:

$$A = Q \underbrace{D_2^{(7)} E_{12}^{(-7)} D_1^{(14)}}_{\text{call this } R}$$

An explicit computation shows that:

$$R = \begin{bmatrix} 14 & -7 \\ 0 & 7 \end{bmatrix}$$

(3) With the matrix Q as on the previous page, consider the linear transformation:

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(\mathbf{v}) = Q\mathbf{v}$$

Suppose you have any two orthogonal (i.e. perpendicular) vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$. Use linear algebra to prove that the vectors $f(\mathbf{v}_1), f(\mathbf{v}_2) \in \mathbb{R}^3$ are also orthogonal. *(5 pts)*

Solution: Because the columns of Q are orthonormal, we have:

$$Q^T Q = I_2$$

In order to have $f(\boldsymbol{v}_1) \perp f(\boldsymbol{v}_2)$, we would need to check that their dot product is 0:

$$(Q\boldsymbol{v}_1)^T Q\boldsymbol{v}_2 = \boldsymbol{v}_1^T \underbrace{Q^T Q}_{I_2} \boldsymbol{v}_2 = \boldsymbol{v}_1^T \boldsymbol{v}_2 = 0$$

where the last equality holds because $\boldsymbol{v}_1 \perp \boldsymbol{v}_2$.

(4) Guess (no explanation needed, just this once) an eigenvector $\mathbf{a} \neq 0$ of the matrix R from the previous page, and the corresponding eigenvalue. Draw the linear transformation:

$$g : \mathbb{R}^2 \to \mathbb{R}^2, \qquad g(\mathbf{w}) = R\mathbf{w}$$

on a picture of $\mathbb{R}^2$, by drawing the eigenvector $\mathbf{a}$ and showing where the function g sends both $\mathbf{a}$ and any other vector in $\mathbb{R}^2$ of your choice, linearly independent from $\mathbf{a}$. *(5 pts)*

Solution: Because R is upper triangular, an eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and the corresponding eigenvalue is 14. We will accept any picture, even if not to precise scale, as long as the coordinates of $g\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and g(the other chosen vector) are clearly marked.

PROBLEM 2

(1) Use row operations (i.e. $\pm$ product of pivots) to compute the determinant of the matrix:

$$\begin{bmatrix} 2 & 3 & 1 \\ -4 & -6 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

(if instead of row operations, you use any other method to compute the determinant, you will lose between 50% and 70% of the points). *(10 pts)*

Solution: Let's perform row reduction:

$$\begin{bmatrix} 2 & 3 & 1 \\ -4 & -6 & -1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{r_2+2r_1} \begin{bmatrix} 2 & 3 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{\text{switch } r_2 \text{ and } r_3} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2-\frac{r_1}{2}} \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

The determinant is the product of the pivots (so $2 \cdot \frac{1}{2} \cdot 1 = 1$) times -1 raised to the number of times we switched rows (so $(-1)^1$). We conclude that $\det = -1$.

(2) Compute $\boxed{z}$ from the system of equations below using Cramer's rule:

$$\begin{bmatrix} 2 & 0 & 1 \\ -3 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

You must explicitly write z as a ratio of determinants, and compute these determinants via cofactor expansion. *(10 pts)*

Solution: Cramer's rule tells us that the solution is:

$$z = \frac{\det \begin{bmatrix} 2 & 0 & 1 \\ -3 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}}{\det \begin{bmatrix} 2 & 0 & 1 \\ -3 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}$$

We will compute the two determinants above by cofactor expansions along the first row:

$$\det \begin{bmatrix} 2 & 0 & 1 \\ -3 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 2 \cdot (-1)^{1+1} \det \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} + 1 \cdot (-1)^{1+3} \det \begin{bmatrix} -3 & -1 \\ 0 & 1 \end{bmatrix} = -7$$

$$\det \begin{bmatrix} 2 & 0 & 1 \\ -3 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = 2 \cdot (-1)^{1+1} \det \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} + 1 \cdot (-1)^{1+3} \det \begin{bmatrix} -3 & -1 \\ 0 & 1 \end{bmatrix} = -5$$

So the solution is:

$$z = \frac{7}{5}$$

(3) Consider an arbitrary $n \times n$ matrix A. We have the following formula for the inverse:

$$\boxed{A^{-1} = \frac{1}{\det A} \cdot C^T}$$

where the (i, j) entry of the cofactor matrix C is $(-1)^{i+j}$ times the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing row i and column j from A.

Find and prove a formula for $\det C$ in terms of $\det A$. **Explain all your steps!** *(5 pts)*

Solution: Since $\det A^{-1} = (\det A)^{-1}$, the identity in the box gives us:

$$(\det A)^{-1} = (\det A)^{-n} \cdot \det C^T \qquad \Rightarrow \qquad \det C^T = (\det A)^{n-1}$$

Since the determinant of a matrix is equal to the determinant of its transpose, we conclude that the answer is $\det C = (\det A)^{n-1}$.

(4) The big formula for the determinant of:

$$\begin{bmatrix} 0 & 0 & 0 & a_{14} & a_{15} & a_{16} \\ 0 & 0 & 0 & 0 & a_{25} & a_{26} \\ 0 & 0 & 0 & 0 & 0 & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ 0 & 0 & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

is a sum of $6! = 720$ terms. Explain why 719 of these terms are zero, but one is non-zero (assuming the a_{ij}'s are non-zero themselves). What is this non-zero term? *(10 pts)*

Solution: The $6! = 720$ terms in the big formula arise as products of various six-tuples of matrix entries, no two on the same row or the same column. So if we want a non-zero term in the big formula, we need to choose 6 non-zero matrix entries, a single one of which lies on any row and any column. Clearly, on the first column we must choose a_{41}. But then we are not allowed to choose any other entry on row 4, so on the second column we must choose a_{52}. But then we are not allowed to choose any other entry on rows 4 and 5, so on the third column we must choose a_{63}. But then we are not allowed to choose any other entry on rows 4, 5 and 6, so on the fourth column we must choose a_{14}. But then we are not allowed to choose any other entry on rows 1, 4, 5 and 6, so on the fifth column we must choose a_{25}. But then we are not allowed to choose any other entry on rows 1,2, 4, 5 and 6, so on the sixth column we must choose a_{36}. We conclude that the only non-zero term in the big formula is:

$$-a_{14}a_{25}a_{36}a_{41}a_{52}a_{63}$$

The $-$ sign in front is due to the fact that we need an odd number of row swaps to put the entries a_{14}, a_{25}, a_{36}, a_{41}, a_{52}, a_{63} in the "usual" top-left to bottom-right ordering of the pivots: you would need to swaps rows 1 and 4, rows 2 and 5, and rows 3 and 6. Alternatively, -1 is the signature of the permutation $(4, 5, 6, 1, 2, 3)$.

PROBLEM 3

Consider the linear transformation $\phi : \mathbb{R}^3 \to \mathbb{R}^3$, $\phi(\mathbf{v}) = A\mathbf{v}$, where:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

We'll give you the following bits of information:

- ϕ fixes a certain (two-dimensional) plane $P \subset \mathbb{R}^3$, i.e. $\phi(\boldsymbol{v}) = \boldsymbol{v}$ for all $\boldsymbol{v} \in P$
- ϕ rescales a certain line $\ell \subset \mathbb{R}^3$ by a factor of 2, i.e. $\phi(\boldsymbol{w}) = 2\boldsymbol{w}$ for all $\boldsymbol{w} \in \ell$

(1) What are the eigenvalues of A, and their algebraic/geometric multiplicities (explain how you know, based on the information given in the bullets above)? *(10 pts)*

Solution: The first bullet tells us that $A\boldsymbol{v} = \boldsymbol{v}$ for all $\boldsymbol{v} \in P$, which means that the eigenvalue 1 has geometric multiplicity at least 2. The second bullet tells us that $A\boldsymbol{w} = 2\boldsymbol{w}$ for all $\boldsymbol{w} \in \ell$, which means that the eigenvalue 2 has geometric multiplicity at least 1. However, the sum of the algebraic multiplicities is 3, so we conclude that the geometric multiplicities are as large as they can be $(2 + 1 = 3)$. Hence the geometric multiplicities are equal to the algebraic multiplicities, and so the eigenvalues are:

$$1, 1, 2$$

(or 1 with algebraic/geometric multiplicity 2 and 2 with algebraic/geometric multiplicity 1).

(2) Compute a basis for the line ℓ in the previous part. *(5 pts)*

Solution: Clearly ℓ is the eigenspace of A corresponding to the eigenvalue 2:

$$\ell = N(A - 2I) = N\left(\begin{bmatrix} 0 & 1 & 1 \\ -1 & -2 & -1 \\ 1 & 1 & 0 \end{bmatrix}\right) \xrightarrow{r_1 \leftrightarrow r_2} N\left(\begin{bmatrix} -1 & -2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\right) \xrightarrow{r_3 + r_1}$$

$$\xrightarrow{r_3 + r_1} N\left(\begin{bmatrix} -1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}\right) \xrightarrow{r_3 + r_2} N\left(\begin{bmatrix} -1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

So a basis of ℓ is given by any vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying:

$$\begin{cases} -x - 2y - z = 0 \\ y + z = 0 \end{cases}$$

By back substitution, such a vector is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Consider the matrix:

$$B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$$

(3) Find an invertible matrix V and a diagonal matrix D such that:

$$\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} = VDV^{-1}$$

(in other words, diagonalize the 2×2 matrix in the left hand side). *(15 pts)*

Hints:

- *It's enough to compute one eigenvalue and its eigenvector (**show all your steps**). Then you can invoke a general principle (**say what it is**) to get the other eigenvalue/eigenvector*
- *Don't be afraid if the answer will involve complex numbers*
- *Don't forget to tell us what V and D are*

Solution: First we compute the characteristic polynomial of $\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$:

$$p(\lambda) = \det \begin{bmatrix} 2-\lambda & -1 \\ 2 & -\lambda \end{bmatrix} = (2-\lambda)(-\lambda) + 2 = \lambda^2 - 2\lambda + 2$$

The roots of the characteristic polynomial are given by the quadratic equation, which gives:

$$\lambda_1 = 1 + i \qquad \text{and} \qquad \lambda_2 = 1 - i$$

As for eigenvectors, we need:

$$\boldsymbol{v}_1 \in N\left(\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} - \lambda_1 I\right) = N\left(\begin{bmatrix} 1-i & -1 \\ 2 & -1-i \end{bmatrix}\right) \xrightarrow{r_1 \leftrightarrow r_2} N\left(\begin{bmatrix} 2 & -1-i \\ 1-i & -1 \end{bmatrix}\right) \xrightarrow{r_2 - \frac{1-i}{2}}$$

$$\xrightarrow{r_2 - \frac{1-i}{2}} N\left(\begin{bmatrix} 2 & -1-i \\ 0 & -1 - \frac{(1-i)(-1-i)}{2} \end{bmatrix}\right) = N\left(\begin{bmatrix} 2 & -1-i \\ 0 & 0 \end{bmatrix}\right)$$

So an eigenvector is given by $\begin{bmatrix} x \\ y \end{bmatrix}$ with $2x - (1+i)y = 0$, such as:

$$\boldsymbol{v}_1 = \begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix}$$

(there are other ways to compute the nullspace above, e.g. you don't need to do the row swap, but then you would need to invert a complex number). As with any matrix with real entries, their complex eigenvalues come in complex conjugate pairs, and the corresponding eigenvectors are also conjugates of each other. Since λ_2 is the conjugate of λ_1, we conclude that an eigenvector of the former is a conjugate of an eigenvector of the latter:

$$\boldsymbol{v}_2 = \begin{bmatrix} \frac{1-i}{2} \\ 1 \end{bmatrix}$$

We conclude that:

$$V = \begin{bmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix}$$